Definitions from Analysis

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Fields: In order for a set to be considered a field, it must have an additive identity (for the sets $\mathbb{Q}, \mathbb{N}, \mathbb{R}, \mathbb{Z}$) this is 0. They must have an additive inverse to define subtraction which in the case of $r \in \mathbb{Q}, \mathbb{Z}, or\ \mathbb{R}$ is -r. Multiplication and division require a multiplicative identity, which is 1 in our sets above. In order for a set to be an *ordered* field, given any two elements $s, r \in A$, it must be true that:

$$r < s,\ r = s,\ or\ r > s$$

Open Sets: A set $O \subseteq \mathbb{R}$ is *open* if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subseteq O$. Think about the endpoints when considering whether a set is open or closed. Obviously if the endpoints are included in the set, then none of that endpoints' ϵ neighborhoods are completely contained in the set. Also consider the density of O in R. If there are gaps or holes in O then it will not be an open set because for a given $a \in O$, you will not be able to find any ϵ-neighborhoods that are entirely in O because you'd be able to find some $x \in \mathbb{R}$ (this is the gap or hole) that is in this neighborhood but not in O.

Limit Points: A point x is a *limit point* of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects the set A in some point other than x. They are also referred to as "cluster points" or "accumulation points". I think this can also be restated as: x is a limit point of A if the intersection of every ϵ neighborhood of x where $\epsilon > 0$ contains a point other than x.

To say that an element is the limit point of some set is to say that it is not isolated, that is, on the Real number line, you can always find a point in the set that is arbitrarily close to the point you're considering as your limit point. This also has to do with density. If your set of interest is dense in $\mathbb{R}$ then every element in your set is surely a limit point. So, this means that every element of $\mathbb{Q}$ is a limit point.

Isolated Points: A point $a \in A$ is an *isolated point* of A if it is not a limit point of A. This is pretty simple: it's the complement of the limit points about the set A. A point in A is isolated if you cannot find another point in A that is arbitrarily close to your chosen point. So, if you have holes on either side of a point, then it should be an isolated point. It's in a sense not connected with the rest of the set. Any element in the natural numbers is isolated because there are large 1 size gaps between any two elements.

Closed Sets: A set $F \subseteq \mathbb{R}$ is *closed* if it contains its limit points. The problem with $\mathbb{Q}$ that prevents it from being a closed set is that $\mathbb{R}$ is dense in $\mathbb{Q}$, which means that between any two rational numbers, you can find a real number (that is not necessarily rational). From that, we can say that picking $a \in \mathbb{R}$ so that $a \notin \mathbb{Q}$ will result in a limit point of $\mathbb{Q}$ because for any distance around a we can find a rational number that is within this distance from a. Because we can find a rational number arbitrarily close to a, we can say that the intersection of all the neighborhoods around a contain an element in $\mathbb{Q}$ which means that a is a limit point of $\mathbb{Q}$ but is not in $\mathbb{Q}$. This means that $\mathbb{Q}$ is not closed.

Closure: Given a set $A \subseteq R$, let L be the set of all limit points of A. The *closure* of A is defined to be $\overline{A} = A \cup L$. Essentially, $\overline{A}$ is the closed version of the set A. If the set A was already closed, then $A = \overline{A}$, otherwise $A \subset \overline{A}$. Moreover, in the case that A is not already closed, it's said that the closure of A, $\overline{A}$ is the smallest closed set containing A.

Compact Sets: A set $K \subseteq \mathbb{R}$ is *compact* if every sequence in K has a subsequence that converges to a limit that is also in K. What's the importance and significance of compactness?

Monotone Convergence Theorem: If a sequence is monotone and bounded, then it converges.

Theorem 2.5.2: Subsequences of a convergent sequence converge to the same limit as the original sequence.

Bolzano-Weierstrass Theorem: Every bounded sequences contains a convergent subsequence.

Cauchy Criterion: A sequence converges if and only if it is a Cauchy sequence.

Nested Interval Property: For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq x_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq ...$$

has a nonempty intersections; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Therorem 3.2.3: (i) The union of an arbitrary collection of open sets is open. (ii) The intersection of a finite collection of open sets is open.

Therorem 3.2.5: A point x is a *limitpoint* of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Therorem 3.2.8: A set $F \subseteq \mathbb{R}$ is closed if and only if every Cauchy sequence contained in F has a limit that is also an element of F.

Uniform Continuity: A function $f : A \mapsto \mathbb{R}$ is uniformly continuous on A if $\forall \epsilon > 0$